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Mathematics School Education provides answers – to which questions?

Abstract

If Mathematics Education at school is the answer, what was the question? What is the primary goal of Mathematics education at schools? My claim will be that (A) it is not one goal but at least three, and (B) these goals are moving targets. To name three primary goals:

1. To present Mathematics as a part of our culture, and as a basis for modern key technologies, which presents answers to very basic, very natural questions, history, present, future.
2. To present Mathematics as a field that equips everyone with the ability to give answers (himself/herself!) to important problems and questions that occur in daily life.
3. To introduce into Mathematics as a field of study – and to lay the foundations for possible (university or vocational) studies, in the Sciences, Engineering, or Mathematics itself.

All these goals change over time – so in shaping and designing the Mathematics school curricula we must be careful that the questions haven't changed fundamentally by the time our answers are being implemented ...

1 My Perspective

If Mathematics Education at school is the answer, what was the question? What is the primary goal of Mathematics education at schools?

I was invited to present here my view on Mathematics school education. In order to make my *view* plausible, I should perhaps first explain my *perspective*.

I am a research mathematician, have been a University professor of mathematics for more than 15 years now. I have received prizes for my research, but I have not worked on education or didactics.Thus, I am looking at Mathematics school education from a university perspective. And I will be talking about the *contents*, not primarily about the *mechanisms* of school education. However, you will see that I believe that in the great panorama of reasons why Mathematics school education fails so often, and to such large extent (and probably in many countries), the *contents* may be a major component.

As a mathematics university professor, teaching (as I did last year, again) basic courses for beginning students, I am confronted with the results of mathematics school education that our students are equipped with. My summary is: **We are not content.** I assume that you are not surprised about this. However, in my view the fact that the students who try to study mathematics at University know too little mathematics as a result of their highschool education is only one component of my and our dissatisfaction. We are multiply not content. Here are **four complaints**.

Complaint I: Insufficient Knowledge

The students we get from school have insufficient knowledge of mathematics.

To exemplify this: At the beginning of my first semester "Linear Algebra" course at TU Berlin last year we did a simple entrance test. Only about 50% of the students, all of whom major in

mathematics, correctly solved a simple exercise with fractions or could produce the formulas for the area and the circumference of a circle of radius r. 84% gave the value of π to two digits after the decimal point. (The correct answer is 3.14.) This shows that clearly many of our mathematics students are not prepared for studying mathematics – or any other scientific subject. And this not only means Berlin highschools don't work: our students come from all over Germany, and also from abroad.

Yes, I know you have heard this complaint before, it is sad, it can be compensated, but it is not the main problem. Here is problem number two:

Complaint II: Insufficient Knowledge Status

The students we get from school have a badly inadequate and insufficient idea about their own state of knowledge.

Many students that I see in exams, oral or written, don't know whether they are good or bad. And they get it wrong in both directions – there are students who think they have mastered it all, and basically haven't understood anything. (In particular, I observe this in self-confident male students.) And I see many students who believe they don't know anything, and really have a firm grasp of all the material. (This occurs not only with female students.)

This is a serious handicap: The students don't know what they know. They do not know how to find out whether they have understood something. They do not know how much they know. As a result they have much too little, or much too much self-confidence.

And it appears that this problem, my complaint number two, got much more serious over the last few decades.

Here comes my most serious and fundamental complaint.

Complaint III: Insufficient Knowledge Framework

The students we get from school have insufficient knowledge about "What is mathematics".

Indeed, let me for this point quote from a 2008 study by three British Sociologists, Heather Mendick, Debbie Epstein and Marie-Pierre Moreau at the "Institute for Policy Studies in Education" at London Metropolitan University. The study was entitled "Maths Images & Identities: Education, Entertainment, Social Justice". It was based on a survery among British students. The authors of the study summarized it as follows:

> Many students and undergraduates seem to think of mathematicians as old, white, middle-class men who are obsessed with their subject, lack social skills and have no personal life outside maths.
>
> The student's views of maths itself included narrow and inaccurate images that are often limited to numbers and basic arithmetic.

The first and the second diagnosis belong together: The mathematicians are part of what is Mathematics!

What is Mathematics? What do you think? Today's school kids may ask *Wikipedia* for help – and be disappointed. Indeed, Wikipedia won't help you on that:

> Mathematics is the study of quantity, structure, space, and change. Mathematicians seek out patterns, formulate new conjectures, and establish truth by rigorous deduction from appropriately chosen axioms and definitions.

Indeed, the German version of Wikipedia goes one step beyond this, and as part of the definition of

Mathematics stresses that there is no commonly accepted definition. I translate:

> Mathematics is the science that developed from the investigation of figures and computing with numbers. For Mathematics, there is no commonly accepted definition; today it is usually described as a science that investigates structures that it created itself for their properties and patterns.

Is this a good answer?

I believe that if you ask education bureaucrats, you will often find the belief that the question "What is Mathematics" is answered by highschool curricula. But what kind of anwer do these give?

If you ask the same question to university mathematicians, they might point you to a very successful book by Richard Courant and Herbert Robbins that has the title "What is Mathematics". However, this is a question – what is the answer? Indeed, the book called "What is mathematics" was first supposed to be called something like "Mathematical Discussions of some basic elementary problems for the general public" – before Thomas Mann convinced Richard Courant that "What is Mathematics" is the title that would sell more copies.

Such investigations could give one an idea about what mathematics is – but is that all?

What is Mathematics? It is at least three things at the same time, which we should consider separately, and to a certain extent also *teach* separately:

1. a collection of **basic tools**, part of everyone's survival kit for modern real life
   – or to put it differently, quoting Dirk Nowitzky's handball coach from the latest issue of the *Mitteilungen*, the *Notices* of the German Mathematical Society "Mathe ist einfach ein saugutes Werkzeug"! – "Maths is a helluva good tool".
2. a **field of knowledge** with a long history, part of our culture, an art, but also very productive, indeed a **production factor**, basis of all modern key technologies, and
3. a highly developed, active, huge **research subject**.

Complaint IV: Insufficient Knowledge of the Activity

The students we get from school have insufficient knowledge about what it means to "do mathematics"

To do mathematics does not mean to compute a number. To do mathematics does not mean to apply a formula. To do mathematics does not mean to find a formula. What does a Mathematician *do*? This is a nice question as a basis of enquiry-based mathematics education in school!

How can Mathematics solve problems? Let me remind you how Mathematicians do *not* solve problems.

1. Typically mathematicians do not *compute a number* that is the answer to your problem, or to any problem. (Indeed, the rumour that the number "42" is the answer to all questions is British humour, which Germans tend to misunderstand.)
2. Typically mathematicians do not just *apply a formula*, or *discover a formula* that solves the problem. (Newspaper stories that start with "Mathematicians have discovered a formula for ..." are always nonsense.)
3. Typically mathematicians do not solve problem in a single passage "from reality to a model that can be solved by mathematics". Indeed in any practical, industrial or even physics situation the process of creating models, adapting parameters, adding constraints, discovering hidden conditions etc. is long, and has to go through many cycles until anything useful will be found. In this process, typically a large amount of paper, pencils, erasers, chalk, computer time, and coffee is used, with little visible effect.

Nevertheless, Mathematics does solve problems, and it contributes knowledge, and it contributes key technologies to virtually all parts of modern high-tech. Indeed, there are large parts of our industry that may be understood as "Mathematical Industries" – industries where mathematical tools are essential for the design, optimization and production. This is not only financial industries, telecommunication industries, and logistics, but nearly all the others as well. Think about it – and tell the kids at school. They have to learn that this is the fact, and that that's what Mathematicians do, and that this is the result of doing mathematics, as part of their view on "What is Mathematics".

As a result of these four deficits, let me formulate my agenda for Mathematics School Education.

2 One Subject called Mathematics is not enough – we need three.

Mathematics I: Basic Tools

Of course, a primary goal of Mathematics Education at school must be to equip *all* pupils with basic mathematics knowledge and abilities. If we are honest, it is not so much Mathematics that we really all need and use in everyday life. But it does include numbers, geometric shapes, probabilites, percentages, and little more than that. However, when have you last solved a quadratic equation in real life? Differentiated a function?

My impression is that this part is the only one that gets any reasonable fraction of space on the school curricula in many countries – but teaching fails miserably, actually for many different reasons, but one of them is lack of motivation, which stems from the fact that kids are not interested in the topic, which is Mathematics I without Mathematics II-III.

Mathematics II: Field of Knowledge

Where does the subject come from? There are 6000 years of mathematics (or even 22000 years) full of stories, of history, of developments, of motivation. Indeed, this part of Mathematics should probably be taught in school in close cooperation or even jointly with Physics and Astronomy, as they are so deeply linked.

The fact that mathematics is not only a set of rules and a finished product, but that it has history is most important for the view of "what is mathematics" – *meet the heroes*! *Stories* about Archimedes, Euler, Gauß, Sonja Kovalevskaya, Andrew Wiles, Grigorij Perelman, Terry Tao or Lisa Sauermann that can shape the image of what mathematics is about!

Still no woman got a fields medal, but indeed four women will be the presidents of four most important world mathematics associations in 2011: among them Ingrid Daubechies, the first woman president of the International Mathematical Union, Marta Sanz Soulé from Barcelona who will be the first woman president of the European Mathematical Society, and Barbara Lee Keyfitz, Ohio State University, who will be the first female president of the International Council for Industrial and Applied Mathematics (ICIAM).

This is also the subject where we can and should connect Mathematics with the other arts! This is where students can *experience* and *feel* mathematics. Mathematics as a subject is alive!

Part of Mathematics as a Field of Knowledge has to be a multitude of *answers to* the question: **What is mathematics good for?** Indeed, many students need these answers as part of their motivation for studying mathematics. Perhaps you are aware of the fact that mathematics is a key component of virtually all modern key technologies. All students have to hear about this. They should also get a chance to get in touch with this, as concretely as possible. Try it out! If possible, on real problems, real data!

Mathematics III: Research Subject

Tell all of them about it! You cannot teach "mathematics research" to all the kids in school, but you have to show them that it exists. That Mathematics is alive, that it is constantly changing. That it is a huge subject, ever expanding! That it encompasses dozens of phantastic areas of studies that you never will hear about at school, such as topology, ergodic theory, measure theory, group theory, Galois theory, Lie theory, *etc.*

Also a part of Mathematics III: **Prepare for University!** That is, provide basics, namely all you need to *know* and to *be able to do* if you want to study (study maths, or any science, or medicine, or any other advanced subject). Clearly this should include the basic *concepts* that will be needed for a successful start into University studies – concepts such as *logic*, *functions*, basic *calculus*, but perhaps more important *proofs*!

Indeed, Mathematics III needs to provide skills for mathematics as a research subject – this heading should thus also contain *proofs*, *problem solving strategies*, and preparation and possibly training for mathematics competitions, – on all different levels, from kangoroo (for *all* the kids) to the International Mathematical Olympiads (for only very few).

Summary: Many Subjects, Moving Targets

To summarize: If we for a moment try to put together Mathematics as a School Subject anew, with a fresh start, then we find that there is a great number of topics – Mathematics School Education must present a **Caleidoscope of Mathematics**:

- Basic Tools
- Field of Knowledge, with Applications
- Research Subject

In the end, question like "which parts of mathematics, facts, components, and skills should we teach to which students – and why" have to be answered:

- Basic Tools are needed by every pupil
- Field of Knowledge – history, stories, applications, overview: important for *motivation*, *education* – also for everyone!
- Research Subject – tell all of them, but preparation for University or other career paths as far as possible/necessary, adapted to level, talent, ambition

Certainly this must be done in a **multitude of ways**: At school there has to be time to

1. explain, practise, memorize,
2. ask questions, search for answers, discover stories,
3. explore, play, and compete.

And as we are talking about a *dynamic* subject, we are indeed talking about **moving targets**:

- Mathematics is constantly changing
- Mathematics School Education has to reflect that.

You won't rewrite Mathematics School Education from scratch, but look to reshape it in view of the picture/answer to "What is Mathematics" it provides.